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Practice Exam 1

MATH 527B
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1. Short computations

(a) If $\varphi \in \mathcal{D}(\mathbb{R})$, show that, in the sense of distributions (in $\mathcal{D}$),

$$\varphi \cdot \delta' = \varphi(0)\delta' - \varphi'(0)\delta$$

Solution. By definitions from class, for all $\psi \in \mathcal{D}$, we have

$$\varphi\delta'(\psi) = \delta'(\varphi\psi) = -\varphi'(0)\psi(0) - \varphi(0)\psi'(0)$$

which yields the desired conclusion. Note, the extra minus comes from integration by parts formula for δ'.

(b) Show that the sequence

$$f_n = \begin{cases} n^2 x^n (1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

converges in the sense of distributions in $C^\infty(\mathbb{R})'$. What is the limit distribution?

Solution. It is easy to check that $f_n(x) \to 0$ for all $x \neq 1$, and further $\int f_n(x)dx = n^2\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{n^2}{(n+1)(n+2)} \to 1$. Thus, we expect that $f_n \to \delta(x-1)$ in distribution. We prove this as follows.

Let $g_n(x) = \int_0^x f_n(t)dt$. Then

$$g_n(x) = \begin{cases} 0 & x \le 0 \\ n^2\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right] & 0 \le x \le 1 \\ \frac{n^2}{(n+1)(n+2)} & x \ge 1 \end{cases}$$

Given $\phi \in C^\infty(\mathbb{R})'$, consider the difference between $\int f_n\phi$ and $\phi(1)$. Clearly

$$\left|\int_{\mathbb{R}} f_n(x)\phi(x)dx - \phi(1)\right| \le \left|\int_{\mathbb{R}} f_n(x)(\phi(x) - \phi(1))dx\right| + \left(1 - \frac{n^2}{(n+1)(n+2)}\right)|\phi(1)|$$

The second term goes to zero as $n \to \infty$. Integrating by parts on the first term, we obtain

$$\begin{aligned} \left|\int_{\mathbb{R}} f_n(x)(\phi(x) - \phi(1))\right| &= \left|g_n(x)(\phi(x)-\phi(1))\Big|_0^1 - \int_0^1 g_n(t)\phi'(t)dt\right| \\ \le \sup_{t\in[0,1]} |\phi'(t)| \int_0^1 g_n(t)dt & \\ &= p_{1,1}(\phi)\left[\frac{n^2}{(n+1)(n+2)} - \frac{n^2}{(n+2)(n+3)}\right] \end{aligned}$$

This also vanishes as $n \to \infty$, so we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\phi(x)dx = \phi(1)$$

for all $\phi \in C^\infty(\mathbb{R})$, so that $f_n(x)$ converges to $\delta(x-1)$ in the sense of distributions in $C^\infty(\mathbb{R})'$.

(c) Show that $T = \sin(x)\mathrm{PV}\left(\frac{1}{x}\right)$ is a distribution on the Schwartz space $\mathcal{S}$ and can be represented by integration, i.e.

$$T(\phi) = \int_{\mathbb{R}} f(x)\phi(x)dx$$

for some continuous function f.

Solution. For $\phi \in \mathcal{S}$, we have

$$\begin{aligned} |T(\phi)| &\leq \int_{|x|>1} \frac{\sin(x)\phi(x)}{x}dx + \int_{-1}^{1} \frac{\sin(x)\phi(x) - 0}{x}dx \\ &\leq q_{1,0}(\phi) \int_{|x|>1} \frac{1}{x^2}dx + 2q_{0,0}(\phi) \\ &\leq 2q_{1,0}(\phi) + 2q_{0,0}(\phi) \end{aligned}$$

where we used the fact that $|\sin(x)| \leq |x|$ for all x and $q_{m,k}$ denotes the usual family of seminorms on $\mathcal{S}$.

It is evident that T is linear, and the above estimate shows that $\phi_j \to 0$ in $\mathcal{S}$ implies that $T(\phi_j) \to 0$, showing that T is a distribution.

If we define $f(x)$ by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

it is easy to see that f is continuous and $T(\phi) = \int_{\mathbb{R}} f(x)\phi(x)dx$ for all $\phi \in \mathcal{S}$.

(d) Let $\rho \in \mathcal{D}'(\mathbb{R}^2)$ be given by $\rho(y^2-1)\delta(x) - 2\delta(x^2+y^2-9)$. Express this distribution in polar coordinates.

Solution. A standard argument, as in the homework shows that, for a smooth function $\phi(x,y)$ and it's expression in polar coordinates given by $\psi(r,\theta) = \phi(r\cos\theta, r\sin\theta)$ we have

$$\delta(x^2+y^2-9)[\phi] = \frac{1}{2}\int_0^{2\pi} \psi(3,\theta)d\theta.$$

Also, in cartesian coordinates

$$\delta(y^2-1)\delta(x) = \frac{1}{2}[\delta(y-1) + \delta(y+1)]\delta(x)$$

so that

$$\delta(y^2-1)\delta(x)[\phi] = \frac{1}{2}(\phi(0,1)+\phi(0,-1))$$

Using this, it is easy to see that, in polar coordinates,

$$\rho[\psi] = 2\psi\big(1,\frac{\pi}{2}\big) + 2\psi\big(1,\frac{3\pi}{2}\big) - \int_0^{2\pi}\psi(3,\theta)d\theta$$

so that

$$\rho = 2\delta(r-1)\delta\big(\theta-\frac{\pi}{2}\big) + 2\delta(r-1)\delta\big(\theta-\frac{3\pi}{2}\big) - \frac{1}{3}\delta(r-3)$$

2. Calculate the distributional derivatives T' and T'' where T is the distribution given by the function

$$f(x) = \begin{cases} Ae^x & -\infty < x \le y \\ Be^{-2x} & y \le x < \infty \end{cases}$$

Using this, or otherwise find a distribution $F_y \in \mathcal{S}'$ such that

$$F_y'' + F_y' - 2F_y = \delta_y' + 3\delta_y$$

Solution. Direct computation, recognizing that $\phi \in \mathcal{S}$ and all of its derivatives vanish faster than any power near infinity yields

$$\begin{aligned} T'(\phi) &= -T(\phi') \\ &= -\int_{-\infty}^{y} Ae^x\phi'(x)dx - \int_y^\infty Be^{-2x}\phi'(x)dx \\ &= (Be^{-2y}-Ae^y)\phi(y) + \int_{-\infty}^{y} Ae^x\phi(x)dx - 2\int_y^\infty Be^{-2x}\phi(x)dx \end{aligned}$$

and

$$\begin{aligned} T''(\phi) &= -T'(\phi') \\ &= -(Be^{-2y}-Ae^y)\phi'(y) - \int_{-\infty}^{y} Ae^x\phi'(x)dx + 2\int_y^\infty Be^{-2x}\phi'(x)dx \\ &= -(Be^{-2y}-Ae^y)\phi'(y) - (Ae^y+2Be^{-2y})\phi(y) \\ &+ \int_{-\infty}^{y} Ae^x\phi(x)dx + 4\int_y^\infty Be^{-2x}\phi(x)dx \end{aligned}$$

Consequently,

$$\begin{aligned} (T''+T'-2T)[\phi] &= -(2Ae^y+Be^{-2y})\phi(y) - (Be^{-2y}-Ae^y)\phi'(y) \\ &= -(2Ae^y+Be^{-2y})\delta_y(\phi) + (Be^{-2y}-Ae^y)\delta_y'(\phi) \end{aligned}$$

Solving the system

$$\begin{aligned} -(2Ae^{y} + Be^{-2y}) &= 3 \\ (Be^{-2y} - Ae^{y}) &= 1 \end{aligned}$$

yields $A = -4/3e^{-y}, B = -1/3e^{2y}$ so that the distribution F_y given by the function

$$F_y(x) = \begin{cases} -\frac{4}{3}e^{x-y} & -\infty < x \leq y \\ -\frac{1}{3}e^{2(y-x)} & y \leq x < \infty \end{cases}$$

satisfies

$$F_y'' + F_y' - 2F_y = \delta_y' + 3\delta_y$$